A $0 - \frac{1}{2}$ law for porosity of measures.

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1 Introduction

In this paper we introduce two definitions of upper porosity of a measure (see Definition 1) which range from 0 to $\frac{1}{2}$ and from 0 to 1 respectively, and prove (Theorem 3) that actually the first upper porosity only can take the extreme values 0 and $\frac{1}{2}$, and the second one takes either the value 0 or the values $\frac{1}{2}$ or 1. The other main result of this paper (see Corollary 4) says that any measure μ which does not satisfy the doubling μ-*a.e.* has a maximal porosity.

1.1 Porosities and the doubling condition.

Let μ be a Radon measure on $\mathbb{R}^n$. Let $B(x,r)$ be the closed ball with center $x \in \mathbb{R}^n$ and radius r. For $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $r > 0$, and $\varepsilon > 0$, let

$$p(A,x,r) := \sup\{\rho : \text{there is a } z \in \mathbb{R}^n \text{ such that } B(z,\rho r) \subset B(x,r)\backslash A\}$$

and

$$\begin{aligned} por(\mu,x,r,\varepsilon) := \sup\{\rho : \; & \text{there is a } z \in \mathbb{R}^n \text{ such that } B(z,\rho r) \subset B(x,r) \\ & \text{and } \mu(B(z,\rho r)) \leq \varepsilon \mu(B(x,r))\}. \end{aligned}$$

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The upper porosity of the set A at the point x is

$$\overline{p}(A,x) = \limsup_{r\downarrow 0} p(A,x,r).$$

For $x \in A$, $\overline{p}(A,x)$ takes values in between 0 and $\frac{1}{2}$. The upper porosity of a set A is given by

$$\overline{p}(A) = \inf\{\overline{p}(A,x) : x \in A\}.$$

The set A is said to be *porous* if $\overline{p}(A) > 0$, it is said to be *strongly porous* if $\overline{p}(A) = \frac{1}{2}$, and σ-strongly porous if A is a countable union of strongly porous sets. Results on porous sets connected with problems in analysis can be seen in [5] and [6].

The upper porosity of μ at x is

$$\overline{por}(\mu,x) := \lim_{\varepsilon\downarrow 0} \limsup_{r\downarrow 0} por(\mu,x,r,\varepsilon).$$

Notice that $\overline{por}(\mu,x)$ takes values in between 0 and 1.

Definition 1 *We introduce the following two definitions $\overline{p}(\mu)$ and $\overline{por}(\mu)$ of upper porosity of μ :*

$$\overline{p}(\mu) := \sup\{\overline{p}(A) : A \subset \mathbb{R}^n \text{ with } \mu(A) > 0\} \text{ and}$$

$$\overline{por}(\mu) := \inf\{s : \overline{por}(\mu,x) \leq s,\ \ \mu\text{-a.e}\, x \in \mathbb{R}^n\}.$$

The definition of $\overline{por}(\mu)$ is in the same spirit that the definition of lower porosity given by J-P. Eckmann, E. Järvenpää and M. Järvenpää in [2]. The same arguments they give in Section 3, allows us to prove that $\overline{p}(\mu) = \overline{por}(\mu)$ for measures μ satisfying the doubling condition μ-*a.e.*

The doubling condition is usually imposed in problems of harmonic analysis, Vitali coverings theorems and tangent measures theory ([1],[3] and [4]). A probability measure μ on $\mathbb{R}^n$ satisfies the doubling condition at a point $a \in \mathbb{R}^n$ if

$$\limsup_{r\downarrow 0} \frac{\mu(B(a,2r))}{\mu(B(a,r))} < \infty.$$

2 Results.

We begin studying the Radon probability measures μ on $\mathbb{R}^n$ which do not satisfy the doubling condition μ-*a.e.* We prove (see Theorem 3) that any Radon probability measure μ gives two alternative decompositions of $\mathbb{R}^n$ into three sets:

- the set where the doubling condition holds, a set with arbitrary small μ-measure and a strongly porous set. This last set is contained in a very sparse set defined as an intersection of disjointed unions of annuli of width tending to zero (see Lemma 2 below).
- the set of points where the doubling condition holds, a set of null μ-measure and a σ-strongly porous set.

The following lemma describes the geometry of the set of points where a measure does not satisfy the doubling condition.

Lemma 2 *Let μ be a Radon probability measure on $\mathbb{R}^n$ and let A be the set of points where μ does not satisfy the doubling condition. Let $\{\lambda_i\}$ be a sequence of real numbers such that $\lim_{i\to\infty}\lambda_i = 1$ and $0 < \lambda_i < 1$, $i \in \mathbb{N}$.. Then for any $\varepsilon > 0$, there exist a family $\{x_{i,j}\}_{i,j\in\mathbb{N}}$ of points in A and a family $\{r_{i,j}\}_{i,j\in\mathbb{N}}$ of radii, with $r_{i,j} < 1/i$ for all $j \in \mathbb{N}$, such that*

$$\mu\left(A\backslash\left(\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{\infty} W_{i,j}\right)\right) \leq \varepsilon$$

where $W_{i,j} := B(x_{i,j}, r_{i,j})\backslash B(x_{i,j}, \lambda_i r_{i,j})$, and for any $i \in \mathbb{N}$ the balls in the family $\{B(x_{i,j}, r_{i,j})\}_{j\in\mathbb{N}}$ are disjointed balls.

This result gives a strong indication that the measures which do not satisfy the doubling condition are exceptional. In particular we conjecture that an ergodic measure invariant for a smooth hyperbolic dynamical system in a n-dimensional manifold must satisfy the doubling condition if its local dimension exceeds $n-1$. We have been unable to prove this conjecture from Lemma 2, which, however, gives easily the following results relating porosity to doubling condition.

Theorem 3 *Let μ be a Radon probability measure on $\mathbb{R}^n$ and let A be the set of points where μ does not satisfy the doubling condition. The following statements hold.*
i) *For all $\varepsilon > 0,$ there is a strongly porous subset A^* of A such that $\mu(A \backslash A^*) \leq \varepsilon$.*
ii) *There exists a σ-strongly porous subset C of A such that $\mu(A) = \mu(C)$.*

Corollary 4 *Let μ be a Radon probability measure on $\mathbb{R}^n$ which does not satisfy the doubling condition μ-a.e. Then $\overline{p}(\mu) = \frac{1}{2}$ and $\overline{por}(\mu) = 1$.*

The next lemma characterizes the measures satisfying the doubling condition μ-*a.e.* and with upper porosity equal to $\frac{1}{2}$, in terms of their tangent measures (see [4]). The set of tangent measures of μ at the point a is denoted by $Tan(\mu, a)$ and the support of the measure μ is denoted by $spt(\mu)$.

Lemma 5 *Let μ be a Radon probability measure on $\mathbb{R}^n$ satisfying the doubling condition μ-a.e. Let $B := \{a \in \mathbb{R}^n$: there is $\nu \in Tan(\mu, a)$ such that $spt(\nu) \neq \mathbb{R}^n\}$. Then*

$$\overline{p}(\mu) = \frac{1}{2} \Longleftrightarrow \mu(B) > 0$$

In order to obtain the first implication of this lemma we only need to assume that $\overline{p}(\mu) > 0$. But if $\mu(B) > 0$ the reverse implication says that $\overline{p}(\mu) = \frac{1}{2}$. If the doubling condition holds μ-*a.e.*, then $\overline{p}(\mu) = \overline{por}(\mu)$ and they only can take the values 0 and $\frac{1}{2}$. Therefore, if the doubling condition does not hold μ-*a.e.* Corollary 4 says that $\overline{p}(\mu) = \frac{1}{2}$ and $\overline{por}(\mu) = 1$. This gives the main result.

Theorem 6 *Let μ be a Radon probability measure on $\mathbb{R}^n$. Then $\overline{p}(\mu)$ is either 0 or $\frac{1}{2}$ and $\overline{por}(\mu)$ is 0, $\frac{1}{2}$ or 1.*

References

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