Lipschitz continuous dynamic programming with discount II

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Abstract

We construct an alternative theoretical framework for stochastic dynamic programming which allows us to replace concavity assumptions with more flexible Lipschitz continuous assumptions. This framework allows us to prove that the value function of stochastic dynamic programming problems with discount is Lipschitz continuous in the presence of nonconcavities in the data of the problem. Our method allows us to treat problems with noninterior optimal paths. We also describe a discretization algorithm for the numerical computation of the value function, and we obtain the rate of convergence of this algorithm.

Keywords: Dynamic programming; nonconcavities; renewable resources; nonsmoothness; increasing marginal returns.

1 Introduction

In this paper we complete the treatment of problems of stochastic dynamic programming with discount in a framework of Lipschitz continuous hypothesis on the data of the problem.

Dynamic programming with discount provides a setting for the analysis of optimal intertemporal transfers of economic resources. There is assumed the existence of a central planner who tries to maximize, over all feasible currents $c_1, c_2, c_3...$ of future consumptions, $\sum_{i=1}^{\infty} \beta^i ER(c_i)$, where $ER(c_i)$ is the expected return at period i derived from consumption c_i and $\beta \in (0,1)$ is the discount factor (see Section 2 for a full exposition of the problem). Typically R is a monetary benefit or some subjective utility which summarizes the central planner's objective, and β reflects the willingness to substitute between present and future return. Some of the principal models in today's Macroeconomic theory as described by Ljungqvist and Sargent [13] are expressible in this framework. Also, many problems at the microeconomic level are currently treated in this setting (see [19]), in particular, problems of optimal exploitation of renewable resources (see Example (10)).

The standard theory of dynamic programming with discount relies heavily on the concavity of the data of the problem (i.e., state space, return function and technological constraint correspondence). It first requires compactness and continuity of the data in order to guarantee the existence, uniqueness and continuity of the value function. Concavity, smoothness and monotonicity are then required in order to guarantee the smoothness and numerical computability of the value function and the optimal policy correspondence. In the deterministic case, these assumptions also guarantee that for discount factors close to 1 the optimal paths converge to an equilibrium state (the so-called turnpike theory). Another standard assumption is to require always interior optimal paths; this allows the recursive computation of the optimal policy through Euler equations. See Stokey et al. [19], Chapters 4, 9, for a detailed analysis of the standard theory.

There are, however, economic and environmental problems that present nonconcavities. Empirical evidence regarding this is given below. See also Maroto and Moran [15] for a discussion of the literature on these problems. The standard assumptions, with the exceptions of compactness and continuity, fail in this setting. The value function can be nonconcave and nonsmooth, and even the numerical analysis lacks a theoretical basis, since no rate of convergence of the algorithms can be obtained from the standard theory.

In Maroto and Moran [15], we construct an alternative theoretical framework based solely on the general hypothesis of Lipschitz continuity of the data. We compute useful information regarding these problems in the case of always interior optimal plans, a case that is relevant in problems of economic growth and in problems of exploitation of renewable resources in which a null or a total consumption is always suboptimal.

There are, however, problems in which the optimal choices for early transitions may be non-interior, whereas the optimal selections are interior at some later transitions. We shall refer such cases as eventually interior optimal plans. Examples of this situation are provided by problems of optimal exploitation of renewable resources in which, due to the presence of increasing marginal returns, it might well be optimal to let the resource grow freely for some periods and then to carry out a large harvesting. In these cases, the optimal policies might be in fact cyclical with periods of null harvesting (see Examples in Section 5).

In this paper we extend the results in Maroto and Moran [15] to the case of eventually interior optimal plans. We establish conditions of Lipschitz continuity on the data (Section 2.2) of a standard discounted dynamic programming problem, in a stochastic setting. Our main result is that in such a setting the value function is Lipschitz continuous (Section 3). This establishes a basis for an analytical study of these problems through the tools of nonsmooth analysis. Our first application of the Lipschitz continuity of the value function of the problem is to show that the discretization algorithm for the computation of the value function derived from this theory converges with a rate $O(\delta)$, with δ being the maximum diameter of the simplices of the discretization net. We then test the robustness of our results via the application of the algorithm to the study of the optimal management of a renewable resource (Section 5). We show that nonconcavities in the data of the problem can lead to conclusions differing dramatically from those of the standard theory. In particular, cycles may exist in the optimal policy dynamics instead of there being a steady state equilibrium.

Research to which the results in this paper can be applied includes studies of the optimal exploitation of *schooling* species, especially clupeids. Bjørndal and Conrad [2] estimated a harvest function for North Sea herring and they found increasing marginal returns (nonconcavities). Similar results were found earlier by Hannesson [10] for the North Atlantic cod fishery. See also references in Examples (Section 5). The schooling species gather in large banks (schools), a behavior which reduces the effectiveness of predators (Partridge [18]). Schooling behavior and the modern fish-finding technology incorporated in fishing vessels, make efficient localization and harvesting of these species possible. This gives rise to a non-concave net revenue function. See Dawid and Kopel [6, 7] for the optimal exploitation of a renewable resource subject to a convex return function.

A second field where the results of this paper find natural application is that of optimal exploitation of renewable resources with a nonconcave growth function that exhibits depensation ("S-shaped"). According to Clark [5], schooling behavior may give rise to such cases. See also Clark [4], Majumdar and Mitra [14], Dechert and Nishimura [8], Le Van and Dana [12], and Olson and Roy [17], for treatment of these problems.

2 Preliminaries

2.1 *The optimization problem*

We describe the stochastic dynamic optimization problem we deal. Let $(X, \mathcal{X})$ be a measurable space with $X \subset \mathbb{R}^n$ and let $\mathcal{X}$ be the $\sigma-$algebra of Borel subsets of X. The space X is assumed to be the domain of an endogenous state variable x, and $Z \subset \mathbb{R}^m$ endowed with the $\sigma-$algebra $\mathcal{B}_m$ of Borel subsets of Z is the domain of a sequence $z_0, z_1, z_2, ...$ of exogenous random shocks. The state of the system at time t is therefore described by a vector (x_t, z_t) taking values in the set $S := X \times Z$. As a topological space, S is endowed with the product topology, and as a measure space it is endowed with the product $\sigma-$algebra. The technological constraints of the problem are represented by a correspondence $\Gamma : S \to X$ which specifies the set $\Gamma(x_t, z_t)$ of feasible states x_{t+1}. We shall write Ω for the graph of Γ, and consider on Ω the topology and $\sigma-$algebra inherited from the product space $X \times X \times Z$.

Let the value z_0 of the first random shock be known, and for $t \geq 1$ assume that the sequence of random variables z_t is a Markov stochastic process with stationary transition function Q, which for $z \in Z$, A in the $\sigma-$algebra $\mathcal{B}_m$ of Borel subsets of $\mathbb{R}^m$ and $t \geq 1$, specifies the conditional probability $Q(z, A)$ that $z_t \in A$ given that $z_{t-1} = z$. This defines in a standard way, for $z_0 \in Z$ and $t \geq 1$, the probability measure $\mu^t(z_0, \cdot)$ on the $t-$fold product space $Z^t = Z \times Z \times ... \times Z$ which specifies the (conditional on z_0) probabilities $\mu^t(z_0, A)$ that the sequence $z^t := (z_1, z_2, ..., z_t)$ of the t first random shocks belongs to the sets A in the

t-fold product $\sigma-$algebra $\mathcal{B}_m^t$.

Given $z_0 \in Z$ and $x_0 \in X$, a planner faces the problem of finding an optimal plan, that is, a constant π_0 and a sequence $\pi_1, \pi_2, \pi_3, ...$of measurable functions $\pi_t := Z^t \to X$ which solves the problem of finding the maximum

$$\sup\{R(x_0, \pi_0, z_0) + \sum_{t=1}^{\infty} \beta^t \int_{Z^t} R(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^t(z_0, dz^t)\}, \quad (1)$$

where the supremum is to be taken over all plans satisfying the technological constraints; $\beta \in (0,1)$ is a discount factor; and R is the return function, defined on the graph Ω of the correspondence Γ, so that $R\ (x_t, x_{t+1}, z_t)$ is the return at time t if the state variable is set to be x_{t+1} at time $t+1$ and the current state of the system is (x_t, z_t).

2.2 *Assumptions*

We describe the conditions required on the data in the above problem. See Maroto and Moran [15] for full details on the scope of application of such conditions, and several related properties. We first introduce some definitions and notations

2.2.1 *Lipschitz functions, correspondences and transition functions*

Given a metric space (Y, d) and a point $x \in Y$, we shall denote by $U(x, r)$ and $B(x, r)$ respectively, the open ball and the closed ball centered at x with radius

r.

Recall that a mapping between two metric spaces $f:(Y,d)\to(Y',d')$ is said to be *Lipschitz* if it satisfies $d'(f(a),f(b))\leq Kd(a,b)$ for all $a,b\in Y$ and some constant K. We shall write $f\in L(K)$ for such a mapping (or $L_A(K)$ if we want to make explicit some subset $A\subset Y$ for which the restriction to A of f is Lipschitz). The function f is said to be *locally Lipschitz on* Y if for every $x\in Y$ there exists a constant K and an open ball $U(x,\varepsilon)$, $\varepsilon>0$, such that $f\in L(K)$ on $U(x,\varepsilon)$, and we write $f\in L^{loc}(K)$ ($L_A^{loc}(K)$) if such condition holds on Y (for f restricted to $A\subset Y$).

A function v will be said to belong to $BL_Y(\alpha,K)$ ($BL_Y^{loc}(\alpha,K)$) if $v\in L_Y(K)$ ($L_Y^{loc}(K)$) and if on Y it is bounded by α (in the supremum norm).

We call $L-$*convex* a subset $C\subset\mathbb{R}^p$ which is a bilipschitz image of a convex set.

The key property of $L-$convex sets is that a function locally Lipschitz on such a set is also globally Lipschitz. The following lemma is easy to prove.

Lemma 1 *Let $C\subset S\subset\mathbb{R}^p$ be an $L-$convex set and f a bilipschitz function such that $f(C)$ is convex, and let v: $S\to\mathbb{R}$ with $v\in L_C^{loc}(K)$. Then $v\in L_C(\Lambda_C K)$ with*

$$\Lambda_C=K_{f^{-1}}K_f, \tag{2}$$

where K_f and $K_{f^{-1}}$ denote respectively the Lipschitz constants of f and f^{-1}.

Remark 2 *If C is convex then we can set $f=id$, which shows $\Lambda_C=1$. It is easy to see that in any case $\Lambda_C\geq1$ (see Maroto and Moran [15]).*

A correspondence $\Gamma : (Y, d) \to (Y', d')$ between metric spaces is said to be a *Lipschitz correspondence* with Lipschitz constant K if $D'_H(\Gamma(a), \Gamma(b)) \leq K D_H(a, b)$ for all $a, b \in Y$, where D'_H and D_H denote the Hausdorff metric on (Y', d') and (Y, d) respectively. We write in this case $\Gamma \in L_Y(K)$.

A compact, non-empty-set valued correspondence $\Gamma : (Y, d) \to (Y', d')$ is said to be *topologically continuous at* $x \in Y$ if it is continuous at x and $D'_H(\partial\Gamma(x), \partial\Gamma(y)) \to 0$ as $y \to x$, where $\partial(\cdot)$ stands for the topological boundary operator. If Γ is continuous at all $x \in Y$, then we say that Γ is a topologically continuous correspondence.

We now define *Lipschitz continuous transition functions.*

To this end we consider the metric d_α in the set $\mathcal{M}_Z$ of Borel probability measure on the space Z of random shocks given by

$$d_\alpha(\mu, \nu) = \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : f \in BL_Z(\alpha, 1) \right\}.$$

For any positive α these are equivalent metrics, and the metric d_1 is called Fortet-Mourier distance (see Dudley [9]). If Z is a compact metric space, it is enough to take the supremum over functions in $L_Z(1)$ in the definition of the metric.

We say that the transition function Q is Lipschitz continuous with Lipschitz constant K, and write $Q \in L_Z(K)$, if $d_\alpha(Q(z_1, \cdot), Q(z_2, \cdot) \leq K d(z_1, z_2)$ holds for all $z_1, z_2 \in Z$, where α is a constant to be specified later (see 4).

We are now ready to formulate our assumptions on the data $X, Z, R, \Gamma,$

and Q. See Maroto and Moran [15] for full details and some straightforward consequences of these assumptions.

Assumption I: *X is a closed subset of R^n and Z is a closed subset of R^m.*

Assumption II: *R is a real bounded function and $R \in L_\Omega(K_R)$ for some positive constant K_R.*

Assumption III: *Γ is a topologically continuous correspondence and $\Gamma \in L_S(K_\Gamma)$ for some positive constant K_Γ.*

Assumption IV: *$Q \in L_Z(K_Q)$ with $K_Q\beta < 1, K_Q > 0$.*

Assumption V: *$O_0 := \{x \in X : G(x,z) \subset int(\Gamma(x,z))$ for all $z \in Z\}$ is a non-empty set. Here G is the optimal policy correspondence.*

We give below the definition of such a correspondence and an extended explanation of this assumption.

Assumption VI :*The domain Z of the random shocks is a compact convex subset of R^m.*

Since X and Z are closed sets by Assumption I, so is S, which, therefore, is also a complete metric space. Let $BC_S(\alpha)$ denote the set of real continuous functions on S bounded in supremum norm by the constant α. The Bellman operator T, defined by

$$T(v(x,z)) = \sup\{R(x,y,z) + \beta \int v(y,\theta)Q(z,d\theta) : y \in \Gamma(x,z)\}, \qquad (3)$$

preserves the set BC_S of real continuous bounded functions on S, i.e. $T : BC_S \to BC_S$, and it is a contractive operator with respect to the supremum norm in BC_S (see Stokey et al. [19], Chapter 9). It is easy to check that if $v \in BC_S(\alpha)$ then $Tv \in BC_S(\| R \| + \beta\alpha)$. Thus, under Assumptions $I - IV$, setting

$$\alpha = \| R \| (1-\beta)^{-1} \tag{4}$$

we have $T : BC_S(\alpha) \to BC_S(\alpha)$.

From now on α will have the value given by (4). There follows from the completeness of $BC_S(\alpha)$ the existence of a unique $V \in BC_S(\alpha)$ such that $T(V) = V$. Moreover, if T^k denotes the $k-$th iterate of T, then $\| T^k(v) - V \| \to 0$ for any $v \in BC_S$, V being the unique value function of the optimization problem (1).

It is well known that the optimal policy correspondence $G : S \to X$, given by

$$G(x,z) = \{y \in \Gamma(x,z) : V(x,z) = R(x,y,z) + \beta \int V(y,\theta) Q(z,d\theta)\},$$

is a compact valued and u.h.c correspondence. For each $v \in BC_S$ a maximizing correspondence G_v may be defined by

$$G_v(x,z) = \{y \in \Gamma(x,z) : Tv(x,z) = R(x,y,z) + \beta \int v(y,\theta) Q(z,d\theta)\}.$$

Observe that, as a consequence of the Theorem of Maximum (see Stokey et al. [19], Theorem 3.6), G_v is a u.h.c. and compact valued correspondence.

Assumptions I to IV are the basic set of assumption for Lipschitz continuous dynamic programming. Assumption V is a rather weak requirement meaning that there are some states x_t of the system from which the optimal state x_{t+1} is not an extremal one. For instance, in the setting of exploitation of renewable resources, this means that the optimal selection for the next period is neither the resource grow freely without consumption nor the total consumption of the resource.

In [15] the case of an always interior optimal plan was analyzed. This case arises when the whole state space considered is contained in the set O_0 described in Assumption V, so that the extremal selections belonging to $\partial\Gamma(x)$ are always suboptimal. This analysis does not require Assumption VI.

In the next section we shall extend our study to the case when the optimal plans are not always interior, but are eventually interior. In the setting of optimal exploitation of renewable resources this means that it might be optimal to let the resource grow freely at some periods, but it will be eventually optimal to carry out a positive consumption which moreover does not exhaust the resource.

3 Lipschitz regularity of the value function

We first state some results obtained in Maroto and Moran [15] that are to be used in the proof of the main result in this section, Theorem 8.

The analysis of the case of always interior optimal plans was based on the study of the Lipschitz constants of the iterates $T^k v$ under the Bellman operator T of functions v in BC_S. This is given by the following lemma

Lemma 3 *Let $v \in BC_S$ and $v \in L(M_0), M_0 \geq 0$. Then, under Assumptions*

$I-$ IV, $Tv \in L(M_1)$ holds, with

$$M_1 = K_R(1 + K_\Gamma) + \max\{1, M_0\}\beta K_Q + M_0\beta K_\Gamma. \qquad (5)$$

See the proof in Maroto and Moran [15].

This lemma shows that the Lipschitz constants M_k of the iterates $T^k v$ satisfy the difference equation

$$M_k = K_R(1 + K_\Gamma) + \max\{1, M_{k-1}\}\beta K_Q + M_{k-1}\beta K_\Gamma. \qquad (6)$$

We shall need below the following specialization of Lemma 3.

Lemma 4 *Let $(x, z), (x', z') \in S$ with $\| (x, z) - (x', z') \| \leq \frac{\rho}{K_\Gamma}$ and assume that, for $v \in BC_S$, $Tv(x, z) \geq Tv(x', z')$ holds. Assume that we only know that $v \in L(M_0)$ on a cylinder $B(y, \rho) \times Z$, where y satisfies*

$$Tv(x, z) = R(x, y, z) + \beta \int v(y, \theta) Q(z, d\theta).$$

Then

$$| Tv(x, z) - Tv(x', z') | \leq M_1 \| (x, z) - (x', z') \|$$

with M_1 as in (5).

Proof. *Since $\Gamma \in L(K_\Gamma)$ we may find an $y' \in \Gamma(x', z')$, with $\| y - y' \| \leq K_\Gamma \| (x, z) - (x', z') \| \leq \rho$, so $y' \in B(y, \rho)$ and, since $(y', \theta) \in B(y, \rho) \times Z$ for all*

$\theta \in Z$, we have

$$v(y,\theta) \leq v(y',\theta) + M_0 K_\Gamma \parallel (x,z) - (x',z') \parallel,$$

and

$$\beta \int v(y,\theta) Q(z,d\theta) \leq \beta \int v(y',\theta) Q(z,d\theta) + M_0 K_\Gamma \beta \parallel (x,z) - (x',z') \parallel,$$

and the proof is completed, since this is exactly the property of v used in the proof of Lemma 4 in [15] (see the chain of inequalities (8)). ■

The next theorem gives the Lipschitz continuity of V in the case of always interior optimal plans. It is the main result of [15]. We recall it here since we use it in the proof of Theorem 8

Theorem 5 *Let $C \subset S$ be a compact set and assume that*

$$G(x,z) \subset int(\Gamma(x,z)) \text{ for all } (x,z) \in C. \tag{7}$$

Let Assumptions $I - IV$ hold. Then $V \in L_C^{loc}(\alpha, K)$, with $K = \max\{1, K_R(1 - \beta K_Q)^{-1}\}$. Let $w \in BL_S(\alpha, M_0)$, for some given constant M_0, and let $\gamma > 0$. Then there exists a $j_0(\gamma)$ such that $T^j w \in BL_C^{loc}(\alpha, K+\gamma)$, $j > j_0(\gamma)$. If C is an $L-$convex set, then $V \in BL_C(\alpha, \Lambda_C K)$ and $T^j w \in BL_C(\alpha, \Lambda_C(K+\gamma))$, all $j > j_0(\gamma)$, with Λ_C given by (2).

We shall also use the following lemma in the proof of Theorem 8.

Lemma 6 *i) If BC_S is endowed with the supremum norm topology and $BC_S \times S$ is endowed with the product topology, then the correspondence $G^* : BC_S \times S \to X$ defined by $G^*(v, c) = G_v(c)$ is upper hemi-continuous.*

ii) Assume that $C \subset S$ is a compact set satisfying (7). There exists an open ball $U(V)$, centered at V, of the normed space BC_S, such that $G_v(x, z) \subset int(\Gamma(x, z))$ holds if $(x, z) \in C$ and $v \in U(V)$.

See proof in Maroto and Moran [15].

This lemma shows that the correspondence G_v (see definition in Section 2.2) satisfies on C the condition required of G in the statement of Theorem 5 for v to be a small perturbation of V.

3.1 *Eventually interior optimal plans*

We are now ready to analyze the Lipschitz regularity of the value function on the set of starting points for optimal plans that may proceed through extremal endogenous states in their initial stages, but which have only interior selections after some future stage. As pointed out in Section 1, this case is relevant in the exploitation of renewable resources in which, due to the presence of increasing marginal returns, it might be optimal to let the resource grow freely for some periods and then to carry out a large harvesting. Optimal cyclical harvesting falls into this case.

Assumptions $I - VI$ are supposed to hold in the remaining of this section

Our first step is the following Lemma

Lemma 7 *O_0 is an open set.*

Proof. Assume that O_0 is not an open set. Then for some $x \in O_0$ there exist sequences $x_n \to x$, $z_n \in Z$ and $y_n \in G(x_n, z_n) \cap \partial(\Gamma(x_n, z_n)) \neq \emptyset$, in which $\partial(\cdot)$ denotes the topological boundary operator. Since Z is a compact set, we may assume $z_n \to z \in Z$ and, therefore, $(x_n, z_n) \to (x, z)$. Using that Γ is topologically continuous we get $\partial(\Gamma(x_n, z_n)) \to \partial(\Gamma(x, z))$ (in the Hausdorff metric). Since $y_n \in \partial(\Gamma(x_n, z_n))$, we see that $d(y_n, \partial(\Gamma(x, z)) \to 0$. By compactness we may choose a subsequence $y_{n_k} \to y \in \partial(\Gamma(x, z))$. This contradicts the upper hemicontinuity of G which demands that $y \in G(x, z) \subset int(\Gamma(x, z))$. ■

We inductively define the sets

$$O_k = \{x \in X : G(x, z) \subset O_{k-1} \text{ for all } z \in Z\}, k = 1, 2, 3...$$

$\overline{Z} : X \to S$, defined by $\overline{Z}(x) = (x, Z)$, is a continuous correspondence (see note 1). Therefore, the correspondence $\overline{G} := G \circ \overline{Z}$ is upper hemi-continuous (see Hildenbrand [11]), and so is $\overline{G}^k$. Since $O_k = \{x : \overline{G}^k(x) \in O_0\}$ (see note 2) we see that $O_k, k = 1, 2, 3,$ are open sets and $O_\omega := (\cup_{k=0}^{\infty} O_k)$ is also an open set. Notice that O_ω consists of the set of endogenous states from which any optimal plan admits only interior selections at some stage.

With these tools in hand, we are ready to state and prove the main result of this section:

Theorem 8 *Let Assumptions $I - VI$ hold. There exist constants $M_k, k = 0, 1, 2, ...$ such that:*

i) If $Y \subset \cup_{j=0}^{k} O_j$ is a compact set and $w \in BL_S(\alpha, M)$, for some given constant M, then there exists a constant $\varepsilon > 0$ and an integer j_0 such that $T^j w$, *for all $j \geq j_0$, and V, belong to $BL(\alpha, M_k)$ on any cylinder $B(x, \varepsilon) \times Z$ with $x \in Y$.*

ii) If Y is an $L-$convex set, then for $j \geq j_0$, $T^j w$ and V belong to $BL_C(\alpha, \Lambda_C M_k)$ with $C = Y \times Z$.

Notice that if $Y \subset O_\omega$ is a compact set, then $Y \subset \cup_{j=0}^{k} O_j$ for some k, so the above theorem applies to any compact set Y with $Y \subset O_\omega$.

Proof. We prove these results by an inductive process.

Let $Y \subset O_0$ be a compact set and $w \in BL_S(\alpha, M)$. Then for every $x \in Y$ there exists an open ball U_x and a closed ball B_x, both centered at x, having the same radius, and contained in O_0. This means that for $(\xi, z) \in B_x \times Z$, $G(\xi, z) \subset int(\Gamma(\xi, z))$ holds. Thus we may use Theorem 5 with $B_x \times Z$ as the compact $L-$convex set to see that there exists a $j_0(x)$ such that $T^j w \in BL(\alpha, M_0)$ for all $j \geq j_0(x)$, on $B_x \times Z$, where $M_0 = K + \gamma$, γ is an arbitrarily small positive constant, and K is as given in the quoted theorem (notice that, since $B_x \times Z$ is a convex set, by Remark 2, $\Lambda_C = 1$ in this case). Since Y is a compact set it may be covered by a collection of balls $\{U_x\}_{x \in Y'}$, where Y' is some finite subset of Y. Hence, $j_0 = \max\{j_0(x) : x \in Y'\}$ and the Lebesgue number ε (see [3]) associated to $\{U_x\}_{x \in Y'}$ satisfy the conditions required in part i) for compact subsets of O_0.

Assume now that part i) has been proven for compact subsets of $\cup_{j=0}^{k-1} O_j$, $k - 1 \geq 0$, for a Lipschitz constant M_{k-1}; assume that Y is a compact subset of $\cup_{j=0}^{k} O_j$; and let $w \in BL_S(\alpha, M)$. Then we can apply our inductive hypothesis to the compact set $Y_1 := Y - O_k \subset \cup_{j=0}^{k-1} O_j$. Let $j_{1,0}$ and ε_1 be the integer and radius given in part i) for the compact set Y_1 and the function w.

Next, consider the compact set $Y_2 = Y - \cup_{\alpha \in Y_1} U(\alpha, \varepsilon_1)$. Since $Y_2 \subset O_k$, we know that $H := \overline{G}(Y_2) \subset O_{k-1}$, and by upper hemi-continuity of $\overline{G}$, H is a compact set (see Hildenbrand [11]). We may also apply to H our inductive hypothesis. Let $j_{2,0}$ and ε_2 be the integer and radius given in part i) for H and w.

We want to use Lemma 3 to obtain (local) Lipschitz constants for the functions $T^j w$ on $Y_2 \times Z$. In the present situation, however, we do not have a Lipschitz constant on all the state space S. We only have such a Lipschitz constant locally on $H \times Z$, so we have to construct carefully neighbourhoods of the points of $Y_2 \times Z$ which land, under $G_{T^j w}$ for j large enough, on neighbourhoods of $H \times Z$ where $T^{j-1} w \in L(M_{k-1})$, and then use Lemma 4.

Let $x \in Y_2$. By upper hemi-continuity of the correspondence $\overline{G^*} : BC_S \times X \to X$, defined by $\overline{G^*}(v, x) := G_v(x, Z)$ (see note 3), we get that there exists an open ball $U^x(V)$ in the metric space BC_S centered at V and an open ball U_x, centered at x, in the metric space X, such that if $(v, \xi) \in U^x(V) \times U_x$, then $G_v(\xi, z) \subset \cup_{\alpha \in H} U(\alpha, \frac{\varepsilon_2}{2})$ for any $z \in Z$. We may suppose, if necessary reducing slightly the radius of U_x, that this also holds for $(v, \xi) \in U^x(V) \times B_x$, where B_x is the closure of U_x. Let $j_{3,0}(x)$ be large enough to guarantee that $v := T^j w \in U^x(V)$ if $j \geq j_{3,0}(x)$.

We shall prove that, for x and v as above and for any $z \in Z$, $Tv \in L(M_k)$ for a suitable M_k on the ball $B_{x,z} := B((x, z), \varepsilon(x))$ with $\varepsilon(x) = \min\{\frac{\varepsilon_2}{4K_\Gamma},$ radius of $B_x\}$. Let $(\xi, \omega), (\xi', \omega') \in B_{x,z}$ and assume that $Tv(\xi, \omega) \geq Tv(\xi', \omega')$

holds. Since $(v,\xi) \in U^x(V) \times B_x$, we may take $y \in G_v(\xi,\omega)$ such that $y \in U(\alpha, \frac{\varepsilon_2}{2})$ for some $\alpha \in H$. Hence $B(y, \frac{\varepsilon_2}{2}) \subset U(\alpha, \varepsilon_2)$. Since $\Gamma \in L(K_\Gamma)$, and using that $\| (\xi,\omega) - (\xi',\omega') \| \leq \frac{\varepsilon_2}{2K_\Gamma}$, we may find $y' \in \Gamma(\xi',\omega') \cap B(y, \frac{\varepsilon_2}{2})$. For $j \geq j_{2,0}$ we know that $v \in L(M_{k-1})$ on $B(y, \frac{\varepsilon_2}{2}) \times Z \subset U(\alpha, \varepsilon_2) \times Z$. So, if $j \geq j(x) := \max\{j_{3,0}(x), j_{2,0}\}$, Lemma 4 applies here if we set $(x,z) = (\xi,\omega)$, $(x',z') = (\xi',\omega')$, and take the ball $B(y, \frac{\varepsilon_2}{2})$ as the ball $B(y,\rho)$. This shows that $T^{j+1}w \in L(M_k)$ with

$$M_k = K_R(1+K_\Gamma) + \max\{1, M_{k-1}\}\beta K_Q + M_{k-1}\beta K_\Gamma$$

on $B_{x,z}$. Therefore $T^{j+1}w \in L^{loc}(M_k)$ on $B(x, \frac{\varepsilon(x)}{2}) \times Z$, $j \geq j(x)$, and since this last set is convex, Lemma 1 gives $T^{j+1}w \in L(M_k)$ on $B(x, \frac{\varepsilon(x)}{2}) \times Z$, for $x \in Y_2$, and $j \geq j(x)$.

Consider now the open cover of Y constituted by the family of balls

$$\{U(x, r(x))\}_{x \in Y_2 \cup Y_1}, \text{ with } r(x) = \frac{\varepsilon(x)}{2} \text{ for } x \in Y_2 \text{ and } r(x) = \varepsilon_1 \text{ for } x \in Y_1.$$

Let $\{U(x, r(x))\}_{x \ \in Y'}$ be a subcover of Y, with Y' some finite subset of Y, and let ε be a Lebesgue number associated to this cover and $j_0 = \max\{j_{1,0}, \max\{j(x) : x \in Y' \cap Y_2\}\} + 1$.

Then, if $j \geq j_0$, $T^j w \in BL(\alpha, M_k)$ on cylinder sets $B(x,\varepsilon) \times Z$, $x \in Y$, which completes the inductive argument. From this it follows that $V \in BL(\alpha, M_k)$ on such cylinders.

We now prove part ii). Notice first that C is an $L-$convex set. To check this, if $f(Y) \subset \mathbb{R}^n$ is a convex bilipschitz image of Y, then $F(C) = f(Y) \times Z$, with $F(x,z) = (f(x), z)$, is a convex bilipschitz deformation of C. Moreover, part i) shows that $T^j w, j \geq j_0$ and V belong to $BL_C^{loc}(M_k)$, and then, by Lemma 1, we get that these functions are also in $BL_C(\Lambda_C M_k)$. ■

4 Numerical analysis

In this section we describe an algorithm for the numerical computation of the value function V on $C = Y \times Z \subset S$, where $Y \subset O_\omega$ is a compact $L-$convex subset of X that supports the optimal policy (see Assumption VII below) and Z is a compact convex subset of $\mathbb{R}^m$. In this situation, C is also compact and $L-$convex.

In order to obtain a rate of convergence for the algorithm we add to the list of assumptions of the previous section the following:

Assumption $VII : G(C) \subset Y$.

This assumption means that in order to make the numerical analysis we must consider a part of the phase space large enough to support the optimal policy correspondence: Assumption VII gives by successive iteration $\overline{G}^k(C) \subset C$ (recall that $\overline{G}(x) = G(x, Z)$).

We know by the previous section (Theorem 8) that $V \in L_C(K_C)$, where K_C is a positive constant and that if $w \in BL_S(\alpha, M)$ then there exists a k_0 and a $\delta_0 > 0$ such that, for $k \geq k_0, T^k w \in L(K_C)$ on cylinders $B(x, \delta_0) \times Z, x \in Y$. In order to obtain a numerical approximation of V on C we discretize the set of points where V is to be approximated by choosing a $\delta-$net N_δ of C with $\delta < \delta_0$, i.e, a subset of C such that for any $c \in C$ there exists some $c' \in N_\delta$ with $d(c, c') \leq \delta$. The compactness of C ensures that N_δ can be taken finite. Analogously we discretize each $\Gamma(x, z)$, $(x, z) \in N_\delta$, through $\delta/4-$nets $\Gamma_{\delta/4}(x, z)$, which allows us to discretize the Bellman operator.

The set of real functions on N_δ will be denoted by $\mathcal{F}_\delta$. Notice that if $\mathcal{F}_\delta$ is endowed with the topology of the supremum norm, then any $w \in \mathcal{F}_\delta$ is a continuous function. According to some rule we define a Borel measurable mapping $\nu : C \to N_\delta$ which fixes a vertex $\nu(x) \in N_\delta$ for each $x \in C$ such that $\nu(x)$ and x always belong to a common simplex $t \in \mathcal{T}_\delta$. This ensures $\| x - \nu(x) \| \leq \delta$.

We shall write $\widetilde{w}$ for the piecewise constant extension to C of a function $w \in \mathcal{F}_\delta$, defined as $\widetilde{w}(c) = w(\nu(c))$ for all $c \in C$.

The discretized Bellman operator is defined on $\mathcal{F}_\delta$ by

$$T_\delta(w(x, z)) = \max_{y \in \Gamma_{\delta/4}(x,z)} \{R(x, y, z) + \beta \int \widetilde{w}(y, \theta) Q(z, d\theta)\}, (x, z) \in N_\delta. \quad (8)$$

It is easy to see that T_δ satisfies Blackwell's conditions (see Stokey et al. [19], Chapter 3) so T_δ is a contractive operator, with contraction factor β, on the space $\mathcal{F}_\delta$ endowed with the supremum norm (which will be denoted by $\| \cdot \|_\delta$). Since N_δ is a compact set, it is a complete metric space. Therefore $\mathcal{F}_\delta$ is a complete metric space, which means that, for $w_0 \in \mathcal{F}_\delta$, the iterates $T_\delta^k(w_0)$ converge to the unique fixed point $V_\delta \in \mathcal{F}_\delta$ of T_δ. Observe that $\widetilde{w}(y, \cdot)$ is a piecewise constant, and it takes constant values on the sets $t_y(\theta) := \{z \in Z : \nu(y,z) = \nu(y,\theta)\}$ (see note 4). These are Borel measurable sets for all $y \in Y$ and $\theta \in Z$ and, given $y \in X$, there is, by compactness of C, a finite collection of them $\mathcal{C}_y := \{t_y(\theta) : \theta \in Z\}$, so that the integral in (8) may be expressed as the finite sum

$$\sum_{t_y(\theta)\in\mathcal{C}_y} \widetilde{w}(y,\theta)Q(z,t_y(\theta)).$$

We shall assume that the probability distributions $Q(z,\cdot),(x,z) \in N_\delta$, are known, or that they can be approximated numerically with arbitrary accuracy. Therefore, the action of the discretized operator T_δ and its iterates T_δ^k can easily be programmed in the form of a computer code, and the fixed point V_δ can be numerically approximated in this way. The next theorem shows that this also allows us to compute the fixed point V of the Bellman operator T (see definition in expression (3)) with arbitrary accuracy on the space C.

Theorem 9 *There exists a constant $\widetilde{K}$ such that, for any sufficiently small δ,*

$$\mid V(x,z) - V_\delta(x,z) \mid < \widetilde{K}\delta \tag{9}$$

holds for every $(x,z) \in N_\delta$. Moreover

$$\mid V(x,z) - V_\delta(\nu(x,z)) \mid < (\widetilde{K} + K_C)\delta \tag{10}$$

holds for any $(x,z) \in C$.

Proof. Let $v_0 \in BL_S(M)$ for some constant M (in practice we may take $v_0 = 0$)

and let $v_k = T^k v_0$. We know that there exists a k_0 such that, if $k \geq k_0$, then $v_k \in L(K_C)$ on any cylinder $B(x, \delta_0) \times Z$ with $x \in Y$ and with a sufficiently small radius δ_0. Set $w_0 = v_{k_0}$ and $w_k = T^k w_0$. Then if $\delta < \delta_0/2$,

$$\mid w_k(c) - w_k(\nu(c)) \mid < K_C \delta \tag{11}$$

holds for all $c \in U_{\delta/2}(Y) \times Z := \bigcup_{x \in Y} U(x, \delta/2) \times Z$ and for all $k \geq 0$.

Let us assume the first assertion proven for $\delta < \delta_0/2$. Using that, for $(x, z) \in C$, $\mid V(x, z) - V(\nu(x, z)) \mid < K_C \delta$ holds if $\delta < \delta_0/2$ and that, by our assumption, $\mid V(\nu(x, z)) - V_\delta(\nu(x, z)) \mid < \widetilde{K} \delta$ holds, we get (10). This shows that if the value function can be approximated with arbitrary accuracy on the grid N_δ, then this also can be accomplished on the whole domain C. Notice that the Lipschitz regularity of V on C plays here a determining role.

We now prove the first assertion in the theorem for $\delta < \delta_0/2$. Let $(x, z) \in N_\delta$ with $\delta < \delta_0/2$ and let $k \geq k_0$. Then, for some $y \in \Gamma(x, z)$,

$$T w_k(x, z) = R(x, y, z) + \beta \int w_k(y, \theta) Q(z, d\theta). \tag{12}$$

Notice that, by Assumption VII, $G^*(V, C) = G(C) \subset Y$ (see Lemma 6 part i for the definition of G^*). By the upper hemicontinuity of G^* (see Lemma 6 part i) we know that there exists an open ball $U(V)$ in the metric space BC_S such that $G^*(U(V) \times C) \subset U_{\delta/4}(Y) := \bigcup_{x \in Y} U(x, \frac{\delta}{4})$. We can assume that k is large enough so that $T^k w_0 = w_k \in U(V)$. Therefore the maximizing y occurring in expression (12) belongs to $U_{\delta/4}(Y)$, and then $(y, \theta) \in U_{\delta/4}(Y) \times Z$ for any

$\theta \in Z$. This, the definitions of T and T_δ, and (11) give

$$\begin{aligned} Tw_k(x,z) &= R(x,y,z) + \beta \int w_k(y,\theta) Q(z,d\theta) \\ &\geq \sup_{y \in \Gamma(x,z)} \{R(x,y,z) + \beta \sum_{t_y(\theta) \in \mathcal{C}_y} \widetilde{w}_k(y,\theta) Q(z, t_y(\theta))\} - K_C\delta \\ &\geq \sup_{y \in \Gamma_{\delta/4}(x,z)} \{R(x,y,z) + \beta \sum_{t_y(\theta) \in \mathcal{C}_y} \widetilde{w}_k(y,\theta) Q(z, t_y(\theta))\} - K_C\delta \\ &= T_\delta(w_k(x,z)) - K_C\delta. \end{aligned}$$

Take $y^* \in \Gamma_{\delta/4}(x,z)$ with $d(y,y^*) < \delta/4$, so that $y^* \in U_{\delta/2}(Y)$. Then

$$\begin{aligned} T_\delta(w_k(x,z)) &\geq R(x,y^*,z) + \beta \sum_{t_{y^*}(\theta) \in \mathcal{C}_{y^*}} \widetilde{w}_k(y^*,\theta) Q(z, t_{y^*}(\theta)) \\ &\geq R(x,y^*,z) + \beta \int w_k(y^*,\theta) Q(z,d\theta) - K_C\delta \\ &\geq R(x,y^*,z) + \beta \int w_k(y,\theta) Q(z,d\theta) - 2K_C\delta \\ &\geq R(x,y,z) + \beta \int w_k(y,\theta) Q(z,d\theta) - (2K_C + K_R)\delta \\ &= T(w_k(x,z)) - (2K_C + K_R)\delta. \end{aligned}$$

From these inequalities we get

$$\mid Tw_k(x,z) - T_\delta w_k(x,z) \mid \leq (K_R + 2K_C)\delta,$$

and for functions $w \in \{w_k\}_{k \geq 0}$ on N_δ,

$$\parallel Tw - T_\delta w \parallel_\delta \leq (K_R + 2K_C)\delta$$

(recall that $\| \cdot \|_\delta$denotes the supremum norm in $\mathcal{F}_\delta$).

Therefore we have for $k > 0$

$$\begin{aligned} \| \quad & T_\delta^k \ w_0 - w_k \ \|_\delta \leq \| \ T_\delta(T_\delta^{k-1} w_0) - T_\delta w_{k-1} \ \|_\delta \qquad (13) \\ + \quad \| \quad & T_\delta w_{k-1} - T w_{k-1} \ \|_\delta \leq \beta \ \| \ T_\delta^{k-1} w_0 - w_{k-1} \ \|_\delta + (K_R + 2K_C)\delta. \end{aligned}$$

Let y_k be the solution sequence, with initial condition $y_0 = 0$, of the first order difference equation

$$y_{k+1} = (K_R + 2K_C)\delta + \beta y_k.$$

Under the convention that T_δ^0 is the identity, $\| \ T_\delta^0 w_0 - w_0 \ \|_\delta = 0 = y_0$, and if $\| \ T_\delta^i \ w_0 - w_i \ \|_\delta \ \leq y_i$ holds for all $i \in \{0, 1, ..., k-1\}$, then, by (13)

$$\begin{aligned} \| \quad & T_\delta^k \ w_0 - w_k \ \|_\delta \leq (K_R + 2K_C)\delta + \beta \ \| \ T_\delta^{k-1} \ w_0 - w_{k-1} \ \|_\delta \\ \leq \quad & (K_R + 2K_C)\delta + \beta y_{k-1} = y_k. \end{aligned}$$

Therefore, $0 \leq \| \ T_\delta^k \ w_0 - w_k \ \|_\delta \leq y_k$ holds for all $k \in \mathbb{N}$. Since $0 < \beta < 1$, the sequence y_k must converge to the equilibrium value of the difference equation, which is given by

$$y_e = (1 - \beta)^{-1}(K_R + 2K_C)\delta.$$

We then may find a k_1 such that $| \ y_e - y_k \ | < \delta$ for $k > k_1$. For such values of k we have

$$0 \leq \| \ T_\delta^k \ w_0 - w_k \ \|_\delta \leq y_k \leq ((1 - \beta)^{-1}(K_R + 2K_C) + 1)\delta.$$

We are now ready to prove the theorem. Let k_2 be large enough to ensure that the inequalities

$$\| \quad V - w_k \|_\delta \leq \delta \text{ and}$$

$$\| \quad V_\delta - T_\delta^k w_0 \|_\delta \leq \delta$$

hold for $k > k_2$, where here V is regarded as a function on N_δ. If $k > \max\{k_0, k_1, k_2\}$ we get

$$\begin{aligned} \| \quad & V - V_\delta \|_\delta \leq \| V - w_k \|_\delta + \| T_\delta^k \ w_0 - w_k \|_\delta + \| V_\delta - T_\delta^k w_0 \|_\delta \\ \leq \quad & ((1-\beta)^{-1}(K_R + 2K_C) + 3)\delta = \widetilde{K}\delta. \end{aligned}$$

■

5 Examples

All data in the examples below were generated using a Compaq AlphaServer GS160 6/731 ALPHAWILDFIRE Computer, coded in standard FORTRAN 77.

5.1 *Deterministic example*

Example 10 ***Optimal exploitation of renewable resources. The case of a concave growth function and convex returns.***

It is assumed that a renewable resource is managed by a sole owner whose objective is to maximize the present value of net revenues derived from the exploitation of the resource

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t R(f(x_t) - x_{t+1}) : \ 0 \leq x_{t+1} \leq f(x_t), \ t = 0, 1., 2... \right\}, \tag{14}$$

where $\beta \in (0,1)$ is a discount factor; x_t is the total biomass at the beginning of period t; $R(h_t)$ is the net revenue function of the sole owner of the resource, where $h_t = f(x_t) - x_{t+1}$ is the harvest in period t; and $f(x_t)$ is the growth function of the biomass in period t.

The Bellman equation associated to (14) is in this case

$$V(x) = \max_{0 \leq y \leq f(x)} \{R(f(x) - y) + \beta V(y)\}. \tag{15}$$

As pointed out in Section 1, the presence of increasing marginal returns in the harvest functions of some schooling species gives rise to nonconcave net revenue functions. We have analyzed the case of the North Sea herring fishery taking into account the presence of increasing marginal returns estimated in the literature (Bjørndal and Conrad [2]). North Sea herring is a schooling species. This species is interesting from an economic and biological point of view (Bjørndal and Conrad [2]).

In order to solve the problem (14), we use parameters based on economic and biological data for the period 1981-2001 (see Nøstbakken and Bjørndal [16] for details on the parameter estimation). For these parameters, the growth function

$f(x_t)$ and the net revenue function $R(h_t)$ are given by the following equations:

$$f(x_t) = x_t + 0.53x_t(1 - x_t), \tag{16}$$

$$R(h_t) = 6.94h_t - 2.544(h_t)^{0.709}. \tag{17}$$

In renewable resources economics it is commonly assumed that the resource is managed by a sole owner whose objective is to maximize the present value of net revenues which is assumed to be concave due to the presence of decreasing marginal returns. This problem can be solved through the standard techniques of discounted dynamic programming theory. The convergence of the optimal paths to an optimal steady state equilibrium is guaranteed for high discount factor levels. When such an equilibrium is attained, the sole owner achieves a stable stock level with a harvest flow sustained ad infinitum. However, due to the presence of increasing marginal returns in the fishery under consideration here, the net revenue function (17) is convex in the harvest, so the standard assumptions of discounted dynamic programming fail in this setting. However, we can apply our algorithm to solve the Bellman equation (15) with net revenue function (17).

In figs. 1 and 2 we show the value function solution of (15) and the associated optimal policy correspondence respectively. In fig. 2 the optimal policy dynamics is also plotted. Figure 2 reveals that the complex discontinuous optimal policy dynamics has one strongly attractive period-five cycle traced for $t = 2001$ from the initial stock level $x_0 = 0.68$. Thus, in contrast to the standard theory, there is not an optimal steady state equilibrium. Rather, the optimal policy dynamics is cyclic. We obtain similar results for high discount factor levels. Therefore, the presence of increasing marginal returns in this fishery leads to conclusions dramatically different from those of the standard theory.

We can also observe in figure 2 that after a big harvest in the first period there are four periods of null harvest (free growth of the resource) due to the fact that the optimal policy correspondence coincides with the growth function of the resource at low stock levels $x \in [0, \bar{x}]$, with $\bar{x} \simeq 0.453$. As a consequence, in this case the optimal policy turns out to be noninterior, while for $x > \bar{x}$, there is a positive harvest for these resource levels. The set O_0 in Theorem 8 where all optimal choices are interior is in this case the open interval $(\bar{x}, 1]$. Therefore the optimal solutions are eventually interior.

5.2 *A stochastic example*

Example 11 *: **Optimal exploitation of renewable resources with random shocks***

We analyze here Example 10 in a stochastic setting. Randomness enters the problem through a multiplicative random shock which modifies the law of growth f, reflecting, for instance, the action of a natural predator on the resource. The intensity z_t of the multiplicative shock at period t is described by an i.i.d. stochastic process $\{z_n\}$ where $z_n = 0,5 + 0,5z'_n$ with z'_n distributed as a $\beta(0.5, 0.5)$. Such distribution is plausible, for instance, for a logistic law of growth of predators, since it is well known that logistic dynamics generates an invariant and asymptotically stable measure with such a distribution (see [1], Chapter 2). The output at period t corresponding to a resource level x is given by $z_t f(x)$ so that the efficiency of the predation ranges from 0% to 50% destruction of the resource. The Bellman equation is written now

$$V(x,z) = \max_{0\leq y\leq zf(x)}\{R(zf(x)-y)+\beta\int_Z V(y,\theta)d\mu(\theta)\},$$

in which $Z = [0.5, 1]$ is the support of the probability distribution μ of z_n. Figure 3 shows the value function of the problem for a discount factor $\beta = 0.95$. In order to obtain a numerical simulation of the optimal policy dynamics we generate 50 random shocks from a $\beta(0.5, 0.5)$. The numerical analysis of 500 simulations reveals that the resource is preserved at low stock levels. In particular, the interval that includes 95% of the resource stock observations is $(0, 0.261]$. Further numerical experiments reveal that the resource becomes extinct for discount factor levels $\beta \in [0.83, 0.94]$. Therefore, in contrast to the standard theory, the presence of nonconcavities in the return function combined with the uncertainty on the growth function of the resource gives rise to the extinction of the resource even for high discount factor levels. Notice that this experiment is in concordance with the empirical evidence on the North Sea herring fishery, which underwent a moratorium in 1977 and careful regulation since 1977, facts that cannot be explained by the standard theory of dynamic programming. However, the aim of this example is not to give a rigorous analysis of the optimal exploitation of this species but to show the analytical capacity of the numerical algorithm. Research in the first direction is in progress.

6 Concluding remarks

In this paper we have completed an alternative theoretical framework which allows us to analyze problems of stochastic dynamic programming with discount in the presence of nonconcavities in the data of the problem. The mathematical complexity inherent in the proofs is compensated by the wide applicability of the results and by the amenability to numerical analysis of the algorithm derived from this theory. This algorithm allows us to analyze numerically problems of optimal exploitation of renewable resources in danger of collapse which are intractable in the standard framework. This algorithm also allow us to explore properties of the optimal policy correspondence related with nonconcavities of the data of the problem, such as countably many points of discontinuity and non-uniqueness following a systematic pattern; discontinuities in the form of jumps upwards in the marginal value function, synchronized with the discontinuities of the optimal policy correspondence; asymptotic cyclic behavior of the optimal paths; existence of a threshold level beyond which there are harvests; and existence of a threshold level below which the resource stock is in danger of collapse even without harvesting. Notice that each of these phenomena is an object of study itself that is potentially analyzable taking the results of this paper as a starting point, and using the available tools of nonsmooth analysis.

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Notes

1.- It is sufficient to observe that $D_H((x, Z), (y, Z)) = d(x, y)$, where D_H is the Hausdorff metric

2.- The integral term $\beta \int v(y, \theta)Q(z, d\theta)$ of Bellman operator forces to use the Lipschitz condition of v in each point of the form (y, θ) for all $\theta \in Z$. This motivates the definition of the sets O_k.

3.- $\overline{G^*}(v, x) = G_v(x, Z) = G_v \circ \overline{Z}(x)$, and the upper hemi-continuity of G_v and $\overline{Z}$ give that of $\overline{G^*}$.

4.- The mapping $t_y : Z \to Z$ may be thought as $t_y(\theta) = pr(\nu^{-1}(y, \theta) \cap (y \times Z)$, where pr is the projection of the fibre $y \times Z$ on Z, given by $pr(y, z) = z$. Since v is a Borel measurable mapping, $\nu^{-1}(y, \theta)$ is a Borel set, and so it is $\nu^{-1}(y, \theta) \cap (y \times Z)$. Lastly, using that pr is a homeomorphism between the metric spaces $y \times Z$ and Z, we see that $t_y(\theta)$ is a Borel set.

Captions

Figure 1. Value function in Example 10 for a discount factor $\beta = 0.83$.

Figure 2. Optimal policy correspondence and optimal dynamics in Example 10 for a discount factor $\beta = 0.83$. Observe the optimal plan, convergent to the atractive period-five cycle traced from the initial state $x_0 = 0.68$.

Figure 3. Value function $V(x, z)$ in Example 11 for a discount factor $\beta = 0.95$ with a multiplicative shock z_t described by a stochastic i.i.d. process $\{z_n\}$ where $z_n = 0,5 + 0,5z'_n$ with z'_n distributed as a $\beta(0.5, 0.5)$.

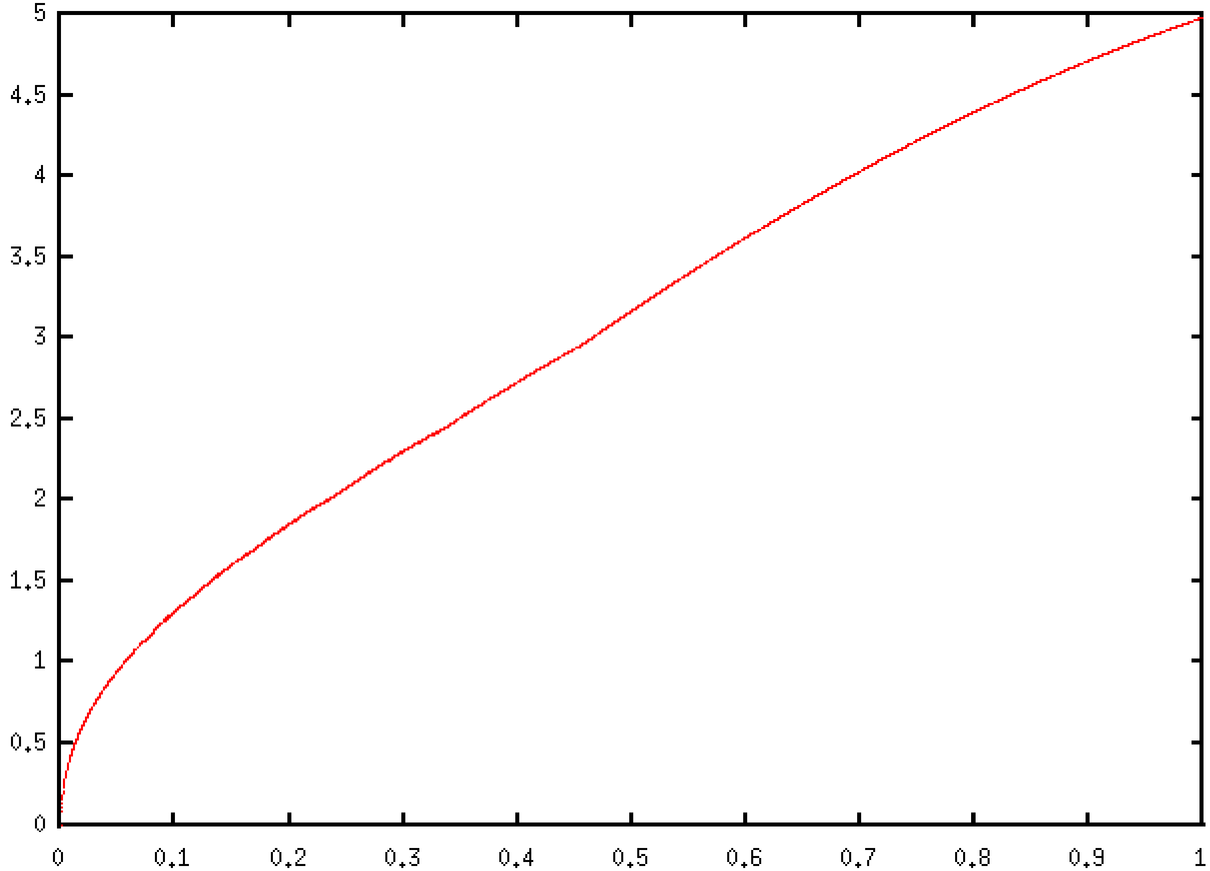

Figure 1. Value function.

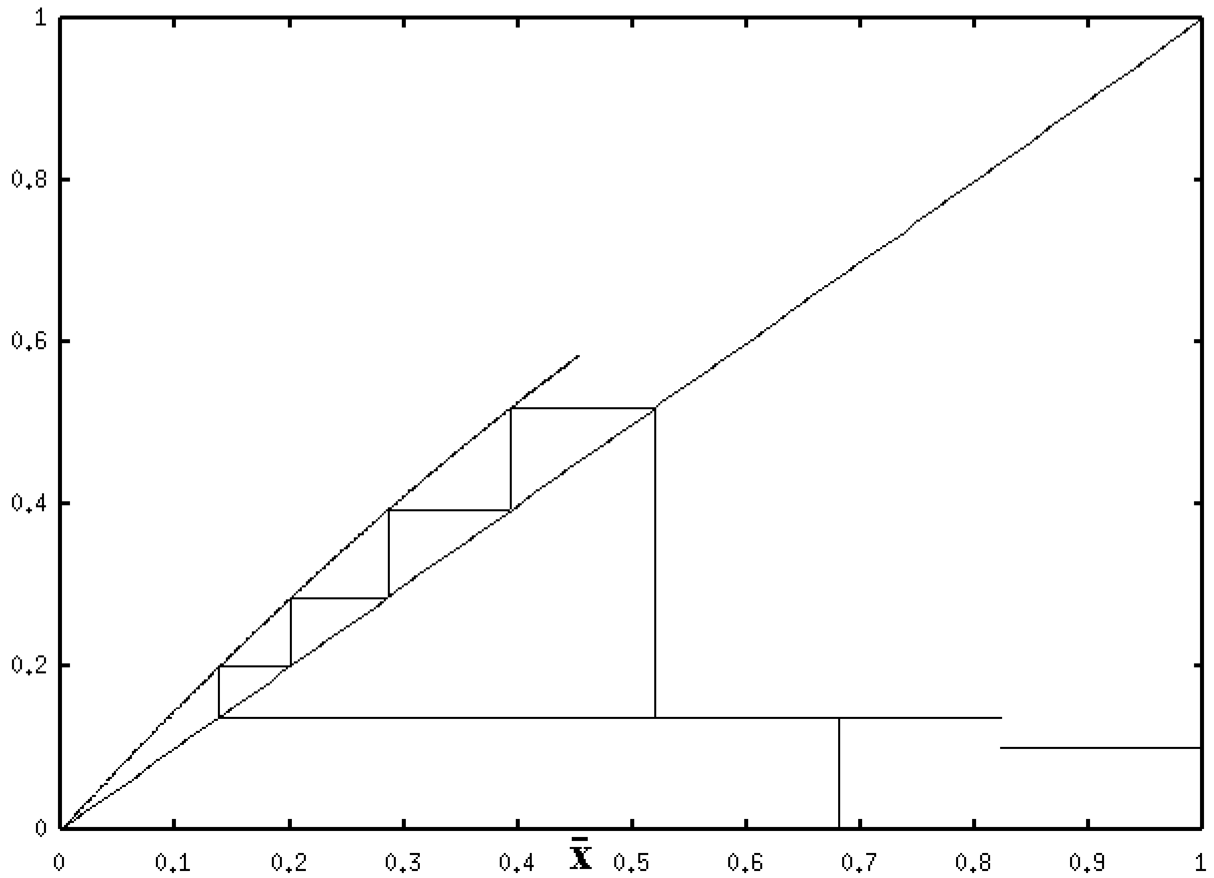

Figure 2. Optimal policy correspondence and optimal dynamics.

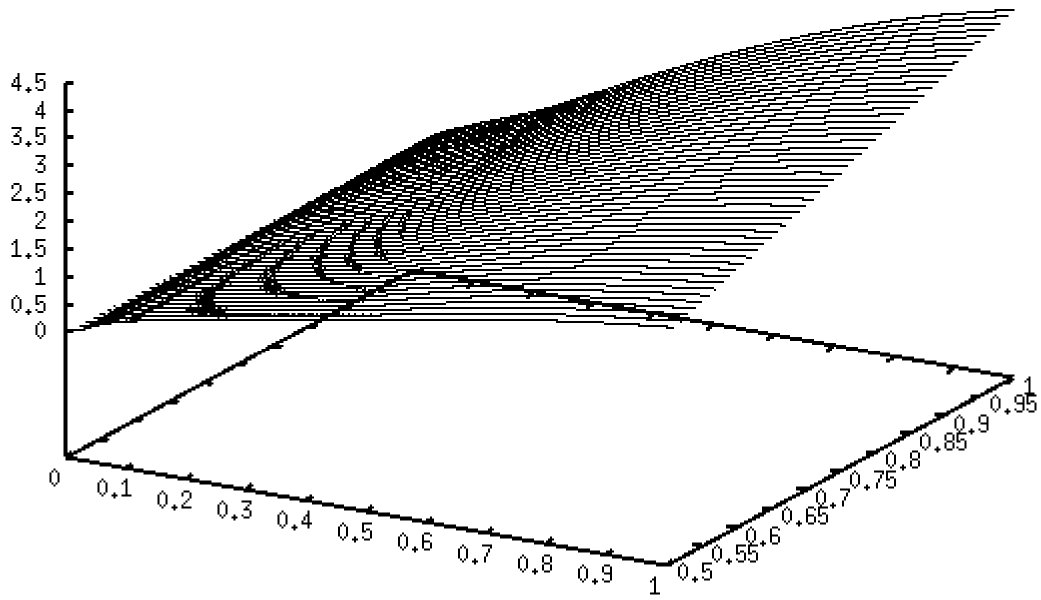

Figure 3. Value function.